Wavelet analysis on the circle

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Matthias Holschneider
Centre de Physique Théorique,[a] *Centre National de la Recherche Scientifique, Luminy, Case 907, 13288 Marseille Cedex 9, France*

The construction of a wavelet analysis over the circle is presented. The spaces of infinitely times differentiable functions, tempered distributions, and square integrable functions over the circle are analyzed by means of the wavelet transform.

I. INTRODUCTION

In this paper we want to show how to analyze fairly arbitrary functions over the circle $\mathbb{T}^1$ with the help of a two-parameter family $g_{\phi,a}$ of functions called wavelets. They are labeled by a position parameter $\phi\in\mathbb{T}^1$ and a scale parameter a, $a>0$. In standard wavelet analysis of functions over the real line $\mathbb{R}$, the family of analyzing wavelets is obtained from a single function by means of dilations and translations (e.g., Ref. 1). On the circle it is difficult to define a good dilation operator, and therefore the wavelets over the circle cannot be obtained by an irreducible representation of the affine group, as was the case in Refs. 1–4.

As proposed before[5] in the case of orthogonal wavelet analysis the wavelets that we will use are obtained from the standard ones $(1/a)g((x-\phi)/a)$ by means of periodization:

$$g_{\phi,a}(x)=\sum_{n\in\mathbb{Z}}\frac{1}{a}g\left(\frac{x-\phi+n}{a}\right),\quad \phi\in\mathbb{T}^1,\quad a\in\mathbb{R}^+. \tag{1.1}$$

This series converges whenever g decays sufficiently fast at infinity. The wavelet transform of a complex-valued function over $\mathbb{T}^1$ is a function over the position–scale space, which is an open, infinite, cylinder $\mathbb{Y}=\mathbb{T}^1\times\mathbb{R}^+$. It is given by the following scalar products:

$$(T_g s)(\phi,a)=(g_{\phi,a},s),\quad (\phi,a)\in\mathbb{Y}. \tag{1.2}$$

The wavelet transform is a sort of mathematical microscope where the position is fixed by the parameter b, the enlargement is $1/a$, and the optic is given by the wavelet itself. We now shall give a precise meaning to all these expressions, and we shall show how to characterize various functional spaces over $\mathbb{T}^1$ with the help of this transform.

II. SOME DEFINITIONS AND EASY PROPERTIES

The space $C^\infty(\mathbb{T}^1)$ is made of complex-valued functions s over $\mathbb{T}^1$ that are arbitrarily many times differentiable. We identify the circle $\mathbb{T}^1$ with the interval $[0,2\pi)$. A topology on $C^\infty(\mathbb{T}^1)$ is given by the following directed family of norms:

$$\|s\|_{C^\infty(\mathbb{T}^1);n}=\sum_{0<p<n}\sup_{\mathbb{T}^1}|\partial^p s|. \tag{2.1}$$

In this topology $C^\infty(\mathbb{T}^1)$ is a Fréchet space, that is, a complete, locally convex, metrizable, linear space. For any function s in $C^\infty(\mathbb{T}^1)$, we can define its Fourier coefficients $(Fs)(n)$, $n\in\mathbb{Z}$:

$$(Fs)(n)=\frac{1}{2\pi}\int_{\mathbb{T}^1}s(x)e^{-inx}\,dx. \tag{2.2}$$

The sequences $(Fs)(n)$ that can appear as the Fourier coefficients of some function s in $C^\infty(\mathbb{T}^1)$ are exactly the sequences that decrease as $|n|$ goes to infinity faster than any power of n. And conversely every such sequence defines a function in $C^\infty(\mathbb{T}^1)$. This sequence space will be called $S(\mathbb{Z})$. A topology on $S(\mathbb{Z})$ is given by the following directed family of norms:

$$\|r\|_{S(\mathbb{Z});n}=\sum_{0<p<n}\sup_{k\in\mathbb{Z}}|k^p r(k)|,\quad n=0,1,\ldots. \tag{2.3}$$

For any sequence r in $S(\mathbb{Z})$ we define the inverse Fourier transform F^{-1}:

$$(F^{-1}r)(x)=\sum_{n\in\mathbb{Z}}r(n)e^{inx}. \tag{2.4}$$

The following well known theorem shows that $C^\infty(\mathbb{T}^1)$ and $S(\mathbb{Z})$ are topologically the same spaces, and that any function in $C^\infty(\mathbb{T}^1)$ can be decomposed into a Fourier series.

Theorem 2.1:
(i) F: $C^\infty(\mathbb{T}^1)\to S(\mathbb{Z})$ is continuous;
(ii) F^{-1}: $S(\mathbb{Z})\to C^\infty(\mathbb{T}^1)$ is continuous,
(iii) F^{-1}: $F=\mathbf{1}_{C^\infty(\mathbb{T}^1)}$, $FF^{-1}=\mathbf{1}_{S(\mathbb{Z})}$.

Proof: [We only shall prove (i) and (ii).] With the help of a partial integration we can write

$$\left|\int_{\mathbb{T}^1}k^p s(x)e^{-ikx}\,dx\right| =\left|\int_{\mathbb{T}^1}\partial_x^p s(x)e^{-ikx}\,dx\right|\leqslant 2\pi\|s\|_{C^\infty(\mathbb{T}^1);p},$$

which proves (i). On the other hand, since any $r\in S(\mathbb{Z})$ is rapidly decreasing, we may exchange the differentiation and the summation:

$$\left|\partial_x^p\sum_{n\in\mathbb{Z}}r(n)e^{inx}\right| =\left|\sum_{n\in\mathbb{Z}}n^p r(n)\frac{1+n^2}{1+n^2}e^{inx}\right| \leqslant(\|r\|_{S(\mathbb{Z});p}+\|r\|_{S(\mathbb{Z});p+2})\sum_{n\in\mathbb{Z}}\frac{1}{1+n^2}\leqslant C\|r\|_{S(\mathbb{Z});p+2}.$$

Q.E.D.

[a] Laboratoire Propre LP-7061, Centre Nationale de la Recherche Scientifique.

This proves (ii). Q.E.D.

The functions g that define via (1.1) the wavelets shall all be in the class $S(\mathbb{R})$ of Schwarz, that is, the set of functions decaying at infinity together with all their derivatives faster than any polynomial. A topology on $S(\mathbb{R})$ is given by the following directed family of norms:

$$\|s\|_{S(\mathbb{R});n,\alpha} = \sum_{\substack{0<p<n \\ 0<1<\alpha}} \sup_{\mathbb{R}} |x^p \partial_x^1 s(x)|, \quad n, \alpha = 0,1,\dots . \tag{2.5}$$

With this topology $S(\mathbb{R})$ is a Fréchet space. On $S(\mathbb{R})$ we define translations and dilations in the usual manner:

$$T^b\colon S(\mathbb{R}) \to S(\mathbb{R}), \quad (T^b s)(x) = s(x-b), \quad b \in \mathbb{R}, \tag{2.6}$$

$$D^a\colon S(\mathbb{R}) \to S(\mathbb{R}), \quad (D^a s)(x) = (1/a) s(x/a), \quad a>0. \tag{2.7}$$

Obviously these operators are continuous. On $S(\mathbb{R})$ we define the Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^{-1}$ as

$$(\mathscr{F}s)(\omega) = \int_{\mathbb{R}} s(x) e^{-i\omega x}\, dx, \tag{2.8}$$

$$(\mathscr{F}^{-1}r)(x) = (2\pi)^{-1} \int_{\mathbb{R}} r(\omega) e^{i\omega x}\, dx. \tag{2.9}$$

We will use the notation $\hat{s}$ for $\mathscr{F} s$. The Fourier transform is a bijective, bicontinuous map.

Theorem 2.2:

(i) $\mathscr{F}\colon S(\mathbb{R}) \to S(\mathbb{R})$ is continuous,

(ii) $\mathscr{F}^{-1}\colon S(\mathbb{R}) \to S(\mathbb{R})$ is continuous,

(iii) $\mathscr{F}^{-1}\mathscr{F} = \mathscr{F}\mathscr{F}^{-1} = \mathbb{1}_{S(\mathbb{R})}$.

For a proof, see any textbook about functional analysis.

The passage from a function s in $S(\mathbb{R})$ to a function in $C^\infty(\mathbb{T}^1)$ will be done by the periodization operator Π:

$$(\Pi s)(x) = \sum_{n\in\mathbb{Z}} s(x + 2\pi n), \quad x \in \mathbb{T}^1. \tag{2.10}$$

Theorem 2.3: $\Pi\colon S(\mathbb{R}) \to C^\infty(\mathbb{T}^1)$ is continuous.

We shall prove this theorem in a moment. To any function in $S(\mathbb{R})$ we can associate a sequence in $S(\mathbb{Z})$ with the help of the sampling operator:

$$\Sigma\colon S(\mathbb{R}) \to S(\mathbb{Z}), \quad (\Sigma s)(n) = s(n), \quad n = \dots, -1,0,1,\dots . \tag{2.11}$$

It obviously is a continuous operator. A natural question is to ask what the Fourier coefficients of periodized function are. The answer is given by the Poisson summation formula, which reads

$$F\Pi = \Sigma\mathscr{F}, \tag{2.12}$$

or, more explicitly ($\hat{s} = \mathscr{F}s$),

$$\sum_{n\in\mathbb{Z}} s(x + 2\pi n) = \sum_{n\in\mathbb{Z}} \hat{s}(n) e^{inx}. \tag{2.13}$$

For a proof of this equation, see, e.g., Ref. 6.

Proof of Theorem 2.3: We have $\Pi = F^{-1}\Sigma\mathscr{F}$. All mappings are continuous. Q.E.D.

The space $L^2(\mathbb{T})$ is made of functions s with finite norm

$$\|s\| = \int_{\mathbb{T}^1} |s(\phi)|^2\, d\phi. \tag{2.14}$$

It is a Hilbert space if it is given the following scalar product:

$$(r,s) = \int_{\mathbb{T}^1} \bar{r}(\phi) s(\phi)\, d\phi. \tag{2.15}$$

Clearly $L^2(\mathbb{T}^1) \supset C^\infty(\mathbb{T}^1)$. The Fourier transform extends to a map from $L^2(\mathbb{T})$ to $L^2(\mathbb{Z})$, the Hilbert space of square summable sequences. We may split $L^2(\mathbb{T})$ into the direct sum of $H^2_-(\mathbb{T}^1)$, the space of functions that have only negative frequencies; $H^2_+(\mathbb{T}^1)$, the space of functions that contain only positive frequencies; and $K(\mathbb{T}^1)$, the constant functions:

$$L^2(\mathbb{T}^1) = H^2_+(\mathbb{T}^1) \oplus K(\mathbb{T}^1) \oplus H^2_-(\mathbb{T}^1). \tag{2.16}$$

The corresponding subspaces of $C^\infty(\mathbb{T}^1)$ shall be denoted by $C^\infty_+(\mathbb{T}^1)$ and $C^\infty_-(\mathbb{T}^1)$.

III. THE WAVELET TRANSFORM OF $C^\infty(\mathbb{T}^1)$

In this section we analyze the space functions with the highest possible regularity. It will turn out that this is mirrored in the wavelet transform by a fast decay of the wavelet coefficients as the scale a goes to 0. The regularity of the analyzing wavelet, or, what is the same, the fast decay of the Fourier transform at infinity, in turn gives rise to a fast decay of the wavelet coefficients, as the scale a goes to infinity. Therefore we will be able to characterize this space as the set of functions that are well localized in the scales; that is, every such function has a minimal effective length scale. We shall characterize the range of the transform, and further give an inversion formula.

First we introduce some notations. For any $f \in S(\mathbb{R})$ and any $a>0$, we define $f_a \in C^\infty(\mathbb{T}^1)$ as

$$f_a(x) = (\Pi D^a f)(x) = \sum_{n\in\mathbb{Z}} \frac{1}{a} f\left(\frac{x + 2\pi n}{a}\right), \tag{3.1}$$

and $f_{\phi,a} \in C^\infty(\mathbb{T}^1)$ with $\phi \in \mathbb{T}^1$ will stand for

$$f_{\phi,a}(x) = (\Pi T^\phi D^a f)(x) = f_a(x - \phi). \tag{3.2}$$

For reasons that will become clear later on we shall require that all the moments of the wavelet, g, vanish:

$$\int_{\mathbb{R}} x^n g(x)\, dx = 0, \quad n = 0,1,\dots . \tag{3.3}$$

An equivalent condition is that the Fourier transform $\hat{g} = \mathscr{F}g$ vanishes at the origin in infinite order:

$$\hat{g}(\omega) = O(\omega^n), \quad n = 0,1,\dots \quad (\omega \to 0). \tag{3.4}$$

The subset of $S(\mathbb{R})$ of functions that satisfy one (and therefore both) of these conditions will be called $S_0(\mathbb{R})$.

Definition 3.1: The wavelet transform T_g of any function $s \in C^\infty(\mathbb{T}^1)$ with respect to a function $g \in S_0(\mathbb{R})$ (called the wavelet) is given by the following scalar products:

$$(T_g s)(\phi,a) = (g_{\phi,a}, s), \quad \phi \in \mathbb{T}^1, \quad a>0. \tag{3.5a}$$

The same expression in Fourier space reads (using the Poisson summation formula)

$$(T_g s)(\phi,a) = \sum_{k\in\mathbb{Z}} (\overline{\mathscr{F}g})(ak) e^{ik\phi} (Fs)(k). \tag{3.5b}$$

The wavelet transform is a function over the position scale space, which in our case is a cylinder $\mathbb{Y} = \mathbb{T}^1 \times \mathbb{R}^+$. Obviously it is a function that is infinitely differentiable. It turns out that T_g is a continuous map from $C^\infty(\mathbb{T}^1)$ into the space of functions $y(\phi,a)$ over $\mathbb{Y}$ that are infinitely differentiable, and that decay for $a\to 0$ and $a\to\infty$ faster than any fractional polynomial in a. We call this space $S(\mathbb{Y})$. The topology of this space is given by a directed family of norms:

$$\|y\|_{S(\mathbb{Y});n,\alpha,\beta} = \sum_{\substack{-n<p<n\\0<l<\alpha\\0<k<\beta}} \sup_{\mathbb{Y}} |a^p\,\partial_a^l\partial_p^k y(\rho,a)|,$$

$$n,\alpha,\beta = 0,1,\ldots. \tag{3.6}$$

Then $S(\mathbb{Y})$ is a Fréchet space. We have the following theorem.

Theorem 3.2: For $g\in S_0(\mathbb{R})$, we have T_g: $C^\infty(\mathbb{T}^1)\to S(\mathbb{Y})$ is continuous.

We first shall prove two lemmas.

Lemma 3.3: Let $s\in S_0(\mathbb{R})$, and let s_a be given by (3.1). Then

(i) $\lim_{a\to 0}\|s_a\|_{L^1(\mathbb{T}^1)} = \|s\|_{L^1(\mathbb{R})}$,

(ii) $\forall n=0,1,\ldots$: $\|s_a\|_{C^\infty(\mathbb{T}^1);n} = O(1/a^m)$
$(a\to\infty)$, for $m=0,1,\ldots$.

Proof: Assertion (i) follows from the fact that for small a essentially only the term $(n=0)$ in the sum (3.1) remains because of the localization of f. To prove (ii) we expand s_a into a Fourier series using the Poisson summation formula:

$$s_a(x) = \sum_{n\in\mathbb{Z}} \hat{s}(an)e^{inx}.$$

Since $\hat{s}(0)=0$ and $\hat{s}(\omega) = O(\omega^{p+m+2})$ as ω goes to infinity, we can estimate, for a large enough,

$$|a^m\,\partial_x^p s_a(x)| \leqslant \sum_{n\neq 0} a^m|n|^p|\hat{s}(a\cdot n)| \leqslant \sum_{n\neq 0}\frac{c}{a^p|n|^{m+2}}.$$

Since the sum remains finite if a goes to infinity we have finished the proof. Q.E.D.

Lemma 3.4: For any $s\in S_0(\mathbb{R})$, there are functions u, v, $w\in S_0(\mathbb{R})$ such that

(i) $\partial_\phi s_{\phi,a}(x) = \partial_x u_{\phi,a}(x)$,

(ii) $\partial_a s_{\phi,a}(x) = \partial_x v_{\phi,a}(x)$,

(iii) $(1/a)s_{\phi,a}(x) = \partial_x w_{\phi,a}(x)$.

Proof: We use again the Poisson summation formula to decompose $s_{\phi,a}$ into a Fourier series $(s=\mathscr{F}s)$:

$$s_{\phi,a}(x) = \sum_{n\in\mathbb{Z}}\hat{s}(an)e^{in(x-\phi)}.$$

The following functions are in $S_0(\mathbb{R})$:

(i) $u=-s$,

(ii) $\hat{u}(\omega) = -i\,\partial_\omega\hat{s}(\omega)$,

(iii) $\hat{w}(\omega) = (i/\omega)\hat{s}(\omega)$.

A direct computation using the Fourier expansion of $s_{\phi,a}$ shows that they satisfy the identities of the lemma.

Proof of Theorem 3.2: First let $p\geqslant 0$. Using Lemma 3.4, for $g\in S_0(\mathbb{R})$ we can find a function $r\in S_0(\mathbb{R})$ such that $\partial_a^l\partial_\phi^k g_{\phi,a}(x) = \partial_x^{l+k}r_{\phi,a}(x)$. With the help of a partial integration we can write

$$|a^p\,\partial_a^l\partial_\phi^k(T_g s)(\phi,a)| = \left|\int_{\mathbb{T}^1} a^p r_{\phi,a}(x)\partial_x^{l+k}s(x)dx\right| \leqslant \sup_{a\in\mathbb{R}^+}\|a^p r_a\|_{L^1(\mathbb{T}^1)}\|s\|_{C^\infty(\mathbb{T}^1);l+k}.$$

Lemma 3.3 shows that the sup is finite. Now let $p<0$. Again Lemma 3.4 can be used to find a function $r\in S_0(\mathbb{R})$ such that $a^p\partial_a^l\partial_\phi^k g_{\phi,a}(x) = \partial_x^{|p|+l+k}r_{\phi,a}(x)$. Therefore we can write

$$|a^p\,\partial_a^l\partial_\phi^k(T_g s)(\phi,a)| = \left|\int_{\mathbb{T}^1} r_{\phi,a}(x)\partial_x^{|p|+l+k}s(x)dx\right| \leqslant \sup_{a\in\mathbb{R}^+}\|r_a\|_{L^1(\mathbb{T}^1)}\|s\|_{C^\infty(\mathbb{T}^1);|p|+l+k}.$$

Again Lemma 3.3 can be used to conclude. Q.E.D.

Definition 3.5: For any function h in $S(\mathbb{Y})$ the inverse wavelet transform $T_g^{-1}h$ is defined as

$$(T_g^{-1}h)(x) = \int_{\mathbb{Y}} g_{\phi,a}(x)h(\phi,a)\,\frac{da\,d\phi}{a}.$$

(We shall see in a moment in which sense T_g^{-1} is the inverse of T_g.) This integral is well defined since h is rapidly decreasing as a tends to 0 or infinity. It again turns out to be a continuous map.

Theorem 3.6: For $g\in S_0(\mathbb{R})$, we have T_g^{-1}: $S(\mathbb{Y})\to C^\infty(\mathbb{T}^1)$ is continuous.

Proof: Since h is rapidly decreasing we may exchange the integration and the differentiation and we may write

$$|\partial_x^n(T_g^{-1}h)(x)| = \left|\int_{\mathbb{Y}}\partial_x^n g_{\phi,a}(x)h(\phi,a)\,\frac{da\,d\phi}{a}\right| = \left|\int_{\mathbb{Y}} g_{\phi,a}(x)\frac{1}{a}\,\partial_x^n h(\phi,a)da\,d\phi\right| \leqslant \int_0^\infty \|g_a\|_{L^1(\mathbb{T}^1)}\,da\cdot\|h\|_{S(\mathbb{Y});1,n,0}.$$

Lemma 3.3 assures that the integral is finite. Q.E.D.

We want to establish the relation between T_g and T_g^{-1}. From Definition 3.1 it follows that the positive (negative) frequencies of the wavelet do only interact with the positive (negative) frequencies of the function s. Therefore we will use the splitting (2.16) to separate positive and negative frequencies. Clearly the image of the constant functions is equal to zero and therefore we can only hope to find an inversion formula that holds on the other two parts.

Theorem 3.7: For all $s\in C^\infty_{+(-)}(\mathbb{T}^1)$ we have

$$T_g^{-1}T_g s = c_g^{+(-)}s.$$

The constants c_g^+ and c_g^- are determined by g:

$$c_g^+ = \int_0^\infty\frac{da}{a}\,|\hat{g}(a)|^2,\quad c_g^- = \int_0^\infty\frac{da}{a}\,|\hat{g}(-a)|^2.$$

Proof: Let $s\in C^\infty_{+(-)}(\mathbb{T}^1)$ be given. Since all negative (positive) frequencies of s vanish, we may suppose that the Fourier transform of g is symmetric. We call s^ε the function that is obtained when using the inverse transformation with a cutoff at the small scales:

$$s^\varepsilon(x) = \int_\varepsilon^\infty \frac{da}{a} \int_{\mathbb{T}^1} d\phi\, g_{\phi,a}(x)(T_g s)(\phi,a)$$
$$= \int_\varepsilon^\infty \frac{da}{a} \int_{\mathbb{T}^1\times\mathbb{T}^1} d\phi\, dy\, g_{\phi,a}(x)\bar{g}_{\phi,a}(y)s(y).$$

For fixed $\varepsilon > 0$, Lemma 3.3 guarantees the absolute convergence of all integrals, and therefore we could exchange the integration. Integrating first over ϕ and a yields

$$s^\varepsilon(x) = \int_{\mathbb{T}^1} K^\varepsilon(x-y)s(y)dy = (K^\varepsilon * s)(x),$$

and the kernel K^ε is given by

$$K^\varepsilon(u) = \int_\varepsilon^\infty \frac{da}{a} \int_{\mathbb{T}^1} d\phi\, g_a(u-\phi)\bar{g}_a(\phi).$$

Expanding these expressions into a Fourier series we obtain the following equations for the Fourier coefficients (denoted by a tilde):

$$\widetilde{K}^\varepsilon(n) = \int_\varepsilon^\infty \frac{da}{a} |\hat{g}(an)|^2 = \int_{|n|\varepsilon}^\infty \frac{da}{a} |\hat{g}(a)|^2 = \widehat{K}(\varepsilon n),$$

where we have posed

$$\widehat{K}(\omega) = \int_{|\omega|}^\infty \frac{da}{a} |\hat{g}(a)|^2.$$

Here we have used the fact that $\hat{g}$ was chosen symmetric. The Fourier coefficients depend only on εn. Applying the Poisson summation formula (2.12), we obtain

$$K^\varepsilon = \Pi D_\varepsilon K = K_\varepsilon, \quad \text{with } K = \mathscr{F}^{-1}\widehat{K}.$$

But $[c_g^{+(-)}]^{-1} K_\varepsilon$ is a summability kernel (e.g., Ref. 6). Therefore we can apply the theorem of the approximation of the identity[6] to conclude. Q.E.D.

We now want to characterize the range of the wavelet transform.

Theorem 3.8: For any $g \in S_0(\mathbb{R})$, the image of $C^\infty(\mathbb{T}^1)$ under T_g is a closed subspace of $S(\Upsilon)$. The range of T_g restricted to $C^\infty_{+(-)}(\mathbb{T}^1)$ consists of exactly those functions y in $S(\Upsilon)$ that satisfy the following "reproducing kernel" equation:

$$y(\phi,a) = \int_\Upsilon p_g(\phi,a;\phi',a')y(\phi',a')\frac{da'\,d\phi'}{a},$$

where the reproducing kernel p_g is given by

$$p_g(\phi,a;\phi',a') = (1/c_g^{+(-)})(g_{\phi,a}, g_{\phi',a'}),$$

or, in Fourier space,

$$p_g(\phi,a;\phi',a') = \frac{1}{c_g^{+(-)}} \sum_{k\in\mathbb{Z}, k>(<)0} (\overline{\mathscr{F}g})(ak)(\mathscr{F}g)(a'k)e^{ik(\phi-\phi')}.$$

Proof: As continuous preimage of a closed space under the inverse wavelet transform, the image of the wavelet transform is closed. Now for any $y \in S(\Upsilon)$ in the range of T_g on $C^\infty_{+(-)}(\mathbb{T}^1)$ there is a function s in $C^\infty_{+(-)}(\mathbb{T}^1)$ such that $T_g s = y$. From Theorem 3.7 we have $T_g^{-1}T_g s = c_g^{+(-)}s$, and therefore we can write

$$T_g T_g^{-1} y = T_g T_g^{-1} T_g s = c_g^{+(-)} T_g s = c_g^{+(-)} y.$$

On the other hand, if $y \in S(\Upsilon)$ satisfies $T_g T_g^{-1} y = c_g^{+(-)} y$, then, for $x = c_g^{+(-)} T_g^{-1} y$, we have $T_g x = y$, which proves that y is in the range of T_g. We now rewrite the above identity more explicitly (we can exchange all integrations, since the integrals are absolutely convergent):

$$\int_\Upsilon \frac{da'\,d\phi'}{a'} y(\phi',a') \int_{\mathbb{T}^1} g_{\phi,a}(x)\bar{g}_{\phi',a'}(x)dx$$
$$= c_g^{+(-)} y(\phi,a),$$

which proves the theorem. Q.E.D.

IV. THE WAVELET TRANSFORM OF $\mathscr{D}'(\mathbb{T}^1)$

In this section we will be interested in the wavelet transform of the "functions" over the circle, with very low regularity, that is, the space of distributions $\mathscr{D}'(\mathbb{T}^1)$. The lack of local smoothness is reflected in the wavelet transform by a polynomial growth of the coefficients at small scale. However, the wavelet transformation allows us to represent any distribution by a C^∞ function over the position scale space.

The elements of $\mathscr{D}'(\mathbb{T}^1)$ are continuous linear functionals of $C^\infty(\mathbb{T}^1)$; that is, for any X in $\mathscr{D}'(\mathbb{T}^1)$ there is an integer number n such that, for any $s \in C^\infty(\mathbb{T}^1)$, we have

$$|X(s)| \leqslant C_{te} \|s\|_{C^\infty(\mathbb{T}^1);n}. \tag{4.1}$$

The smallest such n is called the order of the distribution X. A topology in $\mathscr{D}'(\mathbb{T}^1)$ is given by requiring that any sequence X_n in $\mathscr{D}'(\mathbb{T}^1)$ tends to zero if and only if $X_n(s)$ goes to zero for all $s \in C^\infty(\mathbb{T}^1)$. We identify any function s of $C^\infty(\mathbb{T}^1)$ with the distribution $(s,\cdot)$. This embedding is continuous. The space $C^\infty(\mathbb{T}^1)$ is dense in $\mathscr{D}'(\mathbb{T}^1)$, and therefore any distribution can be approximated by functions in $C^\infty(\mathbb{T}^1)$. We may even choose s_n ($s_n \to X$ in the sense of distributions) in such a way that, for all $r \in C^\infty(\mathbb{T}^1)$ and n, we have (m being the order of X)

$$|(s_n,r)| \leqslant C_{te} \|r\|_{C^\infty(\mathbb{T}^1);m}. \tag{4.2}$$

In the same way we denote by $\mathscr{D}'(\Upsilon)$ the space of linear continuous functionals over $S(\Upsilon)$. For any pair of functions f, h over Υ we define the following "scalar product" $(\cdot,\cdot)_{L^2(\Upsilon)}$ whenever the following integral converges absolutely:

$$(f,g)_{L^2(\Upsilon)} = \int_\Upsilon \bar{f}(\phi,a)g(\phi,a)\frac{da\,d\phi}{a}. \tag{4.3}$$

We now define the wavelet transformation of distributions.

Definition 4.1: T_g: $\mathscr{D}'(\mathbb{T}^1) \to \mathscr{D}'(\Upsilon)$ is defined by $X \in \mathscr{D}'(\mathbb{T}^1) \Rightarrow (T_g X)(y) = X(T_g^{-1}y)$, for all $y \in S(\Upsilon)$.
T_g^{-1}: $\mathscr{D}'(\Upsilon) \to \mathscr{D}'(\mathbb{T}^1)$ is defined by $Y \in \mathscr{D}'(\Upsilon) \Rightarrow (T_g^{-1}Y)(s) = Y(T_g s)$, for all $s \in C^\infty(\mathbb{T}^1)$.

Here, for functions in $C^\infty(\mathbb{T}^1)$ and $S(\Upsilon)$, the definitions of Sec. II apply.

These definitions are reasonable, since T_g and T_g^{-1} are continuous maps between $C^\infty(\mathbb{T}^1)$ and $S(\Upsilon)$. In the following theorem we show that Definition 4.1 actually extends Definitions 3.1 and 3.5.

Theorem 4.2: T_g and T_g^{-1} are the only possible continuous extensions of T_g restricted to $C^\infty(\mathbb{T}^1)$ and T_g^{-1} restricted to $S(\Upsilon)$.

Proof: Let $X_n \in \mathscr{D}'(\mathbb{T}^1)$, $X_n \to 0$ ($n\to\infty$) in the sense of distributions. For all $y \in S(\mathbb{Y})$, we have

$$(T_g X_n)(y) = X_n(T_g^{-1} y) \to 0 \quad (n\to\infty),$$

which shows that T_g is a continuous map from $\mathscr{D}'(\mathbb{T}^1)$ to $\mathscr{D}'(\mathbb{Y})$. In exactly the same way we can show that T_g^{-1} is continuous, too. Let $s \in C^\infty(\mathbb{T}^1)$. Taking s as a distribution in $\mathscr{D}'(\mathbb{T}^1)$, we write, for any $y \in S(\mathbb{Y})$,

$$\begin{aligned}(T_g s)(y) &= \int_{\mathbb{Y}} \frac{da\, d\phi}{a} \int_{\mathbb{T}^1} dx\, \bar{g}_{\phi,a}(x) s(x) y(\phi,a) \\ &= (T_g s, y)_{L^2(\mathbb{Y})}.\end{aligned}$$

We could exchange the integrations since all integrals converge absolutely. Since y was arbitrary, we have proved that Definition 4.1 coincides with Definition 3.1 in the case where a distribution is a function in $C^\infty(\mathbb{T}^1)$. But $C^\infty(\mathbb{T}^1)$ is dense in $\mathscr{D}'(\mathbb{T}^1)$ and therefore by continuity there is exactly one continuous extension. The proof for T_g^{-1} is the same. Q.E.D.

The next theorem shows that the wavelet transform of any distribution in $\mathscr{D}'(\mathbb{T}^1)$ can be identified with a function in $C^\infty(\mathbb{Y})$, that is, the space of functions over $\mathbb{Y}$ that are infinitely differentiable.

Theorem 4.3: Let $X \in \mathscr{D}'(\mathbb{T}^1)$. For all $y \in S(\mathbb{Y})$, we have

$$(T_g X)(y) = (\mathscr{X}, y)_{L^2(\mathbb{Y})},$$

where the function $\mathscr{X}$ is in $C^\infty(\mathbb{Y})$ and is given by $\mathscr{X}(\phi,a) = X(g_{\phi,a})$.

Proof: We can find a sequence s_n of functions in $C^\infty(\mathbb{T}^1)$ that converges to X in the sense of distributions and that satisifies (4.2). Therefore we may write

$$\begin{aligned}(T_g X)(y) &= \lim_{n\to\infty} (s_n, T_g^{-1} y)_{L^2(\mathbb{T}^1)} \\ &= \lim_{n\to\infty} \int_{\mathbb{Y}} \overline{(g_{\phi,a}, s_n)}\, y(\phi,s) \frac{da\, d\phi}{a}. \qquad (4.4)\end{aligned}$$

We could exchange the integration for fixed n because of the absolute convergence of all the integrals. Clearly $(g_{\phi,a}, s_n)$ tends to $X(g_{\phi,a})$ pointwise, that is, for each $(\phi,a) \in \mathbb{Y}$. But from (4.2) it follows that

$$|(g_{\phi,a}, s_n)| \leqslant C \|g_a\|_{C^\infty(\mathbb{T}^1);m},$$

where m is the order of X. Therefore Lemma 3.3 shows that the integrand in (4.4) is uniformly bounded by a function absolutely integrable over $\mathbb{Y}$. We can apply the theorem of dominated convergence to conclude.

Again it is useful to split the whole space into $\mathscr{D}'_{+(-)}(\mathbb{T}^1)$, the space of distributions that are acting on the positive (negative) frequencies only, and $\mathscr{K}(\mathbb{T}^1)$, the distributions that are multiples of the integral over the circle. Again every distribution can be written in a unique way as the superposition of three distributions, each one belonging to one of these three classes:

$$\mathscr{D}'(\mathbb{T}^1) = \mathscr{D}'_+(\mathbb{T}^1) \oplus \mathscr{K}(\mathbb{T}^1) \oplus \mathscr{D}'_-(\mathbb{T}^1).$$

We now will write a distribution in $\mathscr{D}'(\mathbb{T}^1)$ as a well defined "scalar product" of functions in $C^\infty(\mathbb{Y})$ obtained by an absolutely convergent integral over $\mathbb{Y}$.

Theorem 4.4: Let $s \in C^\infty_{+(-)}(\mathbb{T}^1)$ and let X be a distribution in $\mathscr{D}'_{+(-)}(\mathbb{T}^1)$. Then

$$X(s) = (1/c_g^{+(-)})(\mathscr{X}, T_g s)_{L^2(\mathbb{Y})},$$

and $\mathscr{X} = T_g X$ as given by Theorem 4.3.

Proof: From Theorem 3.7 it follows that

$$(T_g X)(T_g s) = X(T_g^{-1} T_g s) = c_g^{+(-)} X(s).$$

Since $T_g s \in S(\mathbb{Y})$, Theorem 4.3 shows how $(T_g X)(T_g s)$ can be written as a scalar product. Q.E.D.

We now want to characterize the image of $\mathscr{D}'(\mathbb{T}^1)$ under the wavelet transform.

Theorem 4.5: The image of $\mathscr{D}'_{+(-)}(\mathbb{T}^1)$ under T_g are exactly those functions $\mathscr{Y}$ in $C^\infty(\mathbb{Y})$ that satisfy (i) there exists $m \in \mathbb{Z}$ such that

$$\mathscr{Y}(\phi,a) = O(1/a^m) \quad (a\to 0)$$

uniformly in ϕ; (ii) for all $p>0$, we have

$$\mathscr{Y}(\phi,a) = O(1/a^p) \quad (a\to\infty)$$

uniformly in ϕ; and (iii) $\mathscr{Y}$ satisfies the reproducing kernel equation (pointwise)

$$\mathscr{Y}(\phi,a) = (1/c_g^{+(-)})(p_g(\phi,a;\cdot,\cdot), \mathscr{Y})_{L^2(\mathbb{Y})}.$$

Proof: Let $X \in \mathscr{D}'_{+(-)}(\mathbb{T}^1)$. Then $\mathscr{Y}(\phi,a) = T_g X = X(g_{\phi,a})$ satisfies (i) and (ii) as follows by direct computation from (4.1) and Lemma 3.3. To show (iii), note that, for any $s \in C^\infty(\mathbb{T}^1)$, we can write, with the help of Theorem 2.4,

$$(T_g^{-1} T_g X)(s) = X(T_g^{-1} T_g s) = c_g^{+(-)} X(s).$$

Therefore, we can write, for $\mathscr{Y} = T_g X$,

$$(T_g T_g^{-1} \mathscr{Y})(\phi,a) = c_g^{+(-)} \mathscr{Y}(\phi,a).$$

On the other hand, a direct computation shows that

$$(T_g T_g^{-1} \mathscr{Y})(\phi,a) = c_g^{+(-)} (p_g(\phi,a;\cdot,\cdot), \mathscr{Y})_{L^2(\mathbb{Y})}.$$

The integral on the right-hand side converges absolutely for every $(\phi,a) \in \mathbb{Y}$ due to the rapid decrease of the reproducing kernel p_g at small scales.

Now suppose that $\mathscr{Y}$ is a locally integrable function over $\mathbb{Y}$ that satisfies (i)–(iii). We define $X \in \mathscr{D}'(\mathbb{T}^1)$ by

$$X(s) = (1/c_g^{+(-)})(\mathscr{Y}, T_g s)_{L^2(\mathbb{Y})}.$$

Clearly X is well defined since $T_g s$ is rapidly decreasing (Theorem 3.2), and a direct computation shows that $T_g X = \mathscr{Y}$, thus showing that $\mathscr{Y}$ is in the image of $\mathscr{D}'_{+(-)}(\mathbb{T}^1)$ under T_g. Q.E.D.

V. THE WAVELET TRANSFORM OF $L^2(\mathbb{T}^1)$

In this section we will analyze the Hilbert space of square integrable functions over the circle. As subspace of $\mathscr{D}'(\mathbb{T}^1)$ all theorems of the previous section hold for $L^2(\mathbb{T}^1)$. In particular, the image of $L^2(\mathbb{T}^1)$ are functions in $C^\infty(\mathbb{Y})$ that satisfy the reproducing kernel equation. It turns out that the wavelet transform is an isometry. Its range is a closed subspace of $L^2(\mathbb{Y})$, the Hilbert space over $\mathbb{Y}$ with scalar product (4.3). So the image of $L^2(\mathbb{T}^1)$ under T_g turns out to be a Hilbert space with reproducing kernel.

Theorem 5.1: The operator

$$(1/\sqrt{c_g^{+(-)}})T_g: \quad H^2_{+(-)}(\mathbb{T}^1) \to L^2(\mathbb{Y})$$

is an isometry. Its adjoint is the only bounded operator form $L^2(\Upsilon)$ to $H^2_{+(-)}(\mathbb{T}^1)$ that coincides with $(1/\sqrt{c_g^{+(-)}})T_g^{-1}$ when it is restricted to $S(\Upsilon)$. Let the wavelet g have only positive (negative) frequency contributions. Then the reproducing kernel gives the orthogonal projector $P_g^{+(-)}$ on the image of $H^2_{+(-)}(\mathbb{T}^1)$ under T_g:

$$P_g^{+(-)}:\ L^2(\Upsilon)\to L^2(\Upsilon),$$

$$(P_g^{+(-)}y)(\phi,a) = (1/c_g^{+(-)})(p_g(\phi,a;\cdot,\cdot),y)_{L^2(\Upsilon)}.$$

Proof: It is enough to show the theorem on a dense subset of $H^2_{+(-)}(\mathbb{T}^1)$. Let $s,u\in C^{\infty}_{+(-)}(\mathbb{T}^1)$. We look upon s as a distribution in $\mathscr{D}'_{+(-)}(\mathbb{T}^1)$. From Theorem 4.4 it follows that

$$(s,u) = (T_g s, T_g u)_{L^2(\Upsilon)},$$

and therefore T_g is an isometry. A direct computation shows that the adjoint is as stated in the theorem. The fact that the reproducing kernel equation is an orthogonal projection operator follows from the well known statement about partial isometries. Q.E.D.

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